WAM 2023 Yearbook

Patterns in Integers: Dynamical and Number Theoretic Approaches

Institute for Advanced Study, Princeton

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This is a compilation of activities and resources contributed by participants during the 2023 Women and Mathematics Program. We hope this can serve as a mathematical and professional reference guide for women mathematicians around the country.

Mathematical Talks

Terng Lectures:

Lillian Pierce, Duke University, "The Circle Method"

Abstract: The series invited participants into the beautiful world of the circle method. This method, which combines both arithmetic and analytic insights, originated 100 years ago in work of Hardy and Ramanujan, in their study of the partition function. It was then more fully developed by Hardy and Littlewood in the study of Waring's problem, which asks how many ways a given integer may be expressed as a sum of s perfect k-th powers. Tthe mechanics of the circle method in the setting of Waring's problem was introduced. Followed by an exploration of the relationship to Ergodic Ramsey Theory by applying the circle method to address questions like the following: must every set of integers with positive density contain a 3-term arithmetic progression? The series concluded by giving a "world tour" of mathematicians currently investigating problems via the circle method.

Terng problem sessions were run by Rena Chu, Duke University.

Uhlenbeck Lectures:

Tamar Ziegler, IAS/Einstein Institute of Mathematics, Hebrew University, "Ergodic Ramsey Theory"

Abstract: A famous theorem of Szemeredi from 1975 states that any subset of positive density in the integers contains arbitrarily long arithmetic progressions. In 1977 Furstenberg gave an ergodic theoretic proof of Szemeredi's theorem. Furstenberg observed that combinatorial statements about patterns in the integers correspond to multiple recurrence questions in ergodic theory. This gave rise to the field of Ergodic Ramsey theory, which centers around proving Ramsey type results using ergodic theoretic techniques (some such results have not alternative proof to this day!). The course introduced the participants to some ideas in ergodic Ramsey theory and also to connections with other approaches to Ramsey type problems including the circle method, which was introduced in the Terng course.

Uhlenbeck Problem Sessions were run by Adi Glücksam, Northwestern University.

References for both courses:

Davenport: Analytic Methods for Diophantine Equations and Diophantine Inequalities

Vaughan: The Hardy-Littlewood Method

Iwaniec and Kowalski: Analytic Number Theory

A. Granville and Z. Rudnick (eds), *Equidistribution in number theory, an introduction*, Proceedings of NATO Advanced Study Institute on Equidistribution in Number Theory, Montreal 2005 Nato Science Series vol 237, Springer 2007.

https://link.springer.com/book/10.1007/978-0-85729-021-2

 https://press.princeton.edu/books/hardcover/9780691642840/recurrence-in-ergodic-theory-and-combinatorial-number-theory

Colloquium

Sarah Peluse, IAS/Princeton University, "Arithmetic Patterns in Dense Sets".

Research Seminar

Maya Sankar, Stanford University
Title: The Turán Number of Homeomorphs

Abstract: A central question in extremal graph theory is to determine ex(n,H), the maximum number of edges in a n-vertex graph containing no subgraph isomorphic to some forbidden graph H. One may define a topological variant of this extremal quantity as follows. Let S be a fixed surface (e.g. the sphere, torus, klein bottle, etc.), and let ex_hom(n,S) denote the maximum number of 2-simplices in an n-vertex simplicial complex in which no sub-complex is homeomorphic to S. I will discuss recent results that completely determine the asymptotics f ex_hom(n,S) for any fixed surface S, as well as some techniques for bounding ex_hom(n,X) for arbitrary 2-dimensional simplicial complexes X.

Kathryn Wilson, Kansas State University
Title: Counting Solutions to Diophantine Inequalities in Function Fields

Abstract: I will discuss how the Bentkus-Götze-Freeman variant of the Davenport-Heilbronn circle method can be used to study $F_q[t]$ solutions to inequalities of the form

$$\mathrm{ord}(\lambda_1 x_{k_1} + \cdots + \lambda_s x_{k_s} - \tau) < \epsilon,$$

where constants $\lambda 1, \cdots, \lambda s, \tau \in \mathrm{F}q((1/t))$ satisfy certain conditions. After introducing some function field notation, I will give a sketch of how to count $\mathrm{F}q[t]$ solutions in this setting.

Rena Chu, Duke University
Title: Exponential sums with applications in PDEs

Abstract: In 2016 Bourgain applied Gauss sums to construct a counterexample related to a decades-old question in PDEs. The story started in 1980 when Carleson asked about how "smooth" an initial data function must be to imply pointwise convergence for the solution of the linear Schr\"odinger equation. After progress by many authors, this was resolved by Bourgain, whose counterexample construction proved a necessary condition on the regularity, and Du and Zhang, who proved a sufficient condition. Bourgain's methods were number-theoretic, and this raised a natural question: could number-theoretic properties of other exponential sums have implications for other dispersive PDEs? We developed a flexible new method to construct counterexamples for analogues of Carleson's question. In particular, this applies the Weil bound for exponential sums, a consequence of the truth of the Riemann Hypothesis over finite fields.

Jessica Liu, CUNY
Title: Dynamical Properties of the Indicator Sequence of Square-Free Numbers

Abstract: Every infinite sequence on a finite set of symbols gives rise to a dynamical system by taking the topological closure of the set of iterates of the sequence under the shift map, which deletes the first symbol of a sequence. Given any dynamical system, one is often interested in different notions of entropy, which are quantities that reflect the "amount of information" in the system. The indicator sequence of square-free numbers is the sequence that is 1 if the index is not divisible by a square number and 0 otherwise. After explaining the relevant definitions and constructions, we will outline a short proof of a result by Peckner that there is a unique measure such that the measure-theoretic entropy of this system realizes the topological entropy, which is different from Peckner's original proof.

Adi Glücksam, Northwestern University
Title: Unbounded fast escaping wandering domains

Abstract: Complex dynamics explores the evolution of points under iteration of functions of complex variables. In this talk I will describe an interesting phenomenon in complex dynamics called wandering domains. I will mention a few techniques used to construct examples and describe what is known and what is unknown. This short-talk is based on a joint work with V. Evdoridou, and L. Pardo-Simón.

Katharine Woo, Princeton University
Title: Small scale distribution of primes in four-term arithmetic progressions

Abstract: In 1985, Maier demonstrated that there are short intervals with an exceptionally large or small number of primes. In this talk, I will discuss an analog of this result for primes in three-term and four-term arithmetic progressions. I aim to highlight the similarities and differences between the proofs of the two cases; the former uses the circle method, whereas the latter requires ergodic theory. This is based on joint work with Mayank Pandey.

WAM Ambassadors

More information on how to apply for the WAM Ambassadorships please go here:
https://www.ias.edu/math/wam/apply/ambassador-program

For the reports from the Ambassadorships for 2022-2023 please go here to watch the video summaries of the events: https://www.ias.edu/math/wam/program-years/2023-program-women-and-mathematics/2023-ambassador-reports